ON WCG ASPLUND SPACES AND EBERLEIN COMPACTA

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INTRODUCTION

In [7], Namioka asked conditions on a RadonNikodym (RN) com pact to be Eberlein. In [9], Orihuela, Schachermayer and Valdivia, showed that, a RN compact that were also Corson compact had to be Eberlein (and after Gul'ko's result, every RN and Gul'ko compact is also Eberlein). In the same line, conditions on an Asplund space to be WCG are given by Fabian, Valdivia and others (see [1] Th. 8.3.3).

MAIN RESULTS

We are going to give some results where the following notion, introduced in [4], will play a very important role.

Definition 1. *Let X be a set; τ_1 and τ_2 two topologies defined on X. We shall say that X has property $\mathcal{L}(\tau_1, \tau_2)$ if, for every $x \in X$ there exists a countable set S(x), containing x, so that if $A \subset X$, then $\overline{A}^{\tau_2} \subset \overline{\cup\{S(x); x \in A\}}^{\tau_1}$.*

In [5] and [6] the authors show that when both topologies are metrizable, $\mathcal{L}(d_1, d_2)$ is equivalent to (X, d_2) being d_1 SLD. Which is a notion introduced by Jayne, Namioka and Rogers in [3].

Definition 2. *Let (X, τ) be a topological space and let d be a metric on X. It is said that X has a countable cover by set of small local diameter (SLD) if for every $\varepsilon > 0$ there exists a decomposition $X = \cup_{n=1}^{\infty} X_n^{\varepsilon}$ such that for each $n \in \mathbb{N}$ every point of X_n^{ε} has a relatively τneighbourhood of ddiameter less than ε.*

A variant of the LSP for topological vector spaces is the following:

Definition 3. *Let A be a subset of a topological vector space $X; \tau_1$ be a topology defined on X and τ_2 a topology defined on A. We shall say that A has property $span\mathcal{L}(\tau_1, \tau_2)$ if, for every $x \in X$ there exists a countable set $S(x) \subset A$, containing x, so that if $B \subset A$, then $\overline{B}^{\tau_2} \subset \overline{span \cup \{S(x); x \in B\}}^{\tau_1}$.*

In [5] it was shown that on a vector space X, for two metrics, having $\mathcal{L}(d_1, d_2)$ is equivalent to having $span\mathcal{L}(d_1, d_2)$.

Definition 4. *Let (X, τ) be a topological space. We shall say that (X, τ) has the Linking Separability Property (LSP), if there exists a metric d, finer than τ, such that X has $\mathcal{L}(d, \tau)$. Analogously we define spanLSP.*

We shall say that a Banach space has LSP when $(X, weak)$ has LSP. Let $(X, weak)$ be a Banach space and d a metric finer than the weak topology. Hansell showed in [2] that if the metric d has a network wich is σisolated in $(X, weak)$, then so has the norm topology. Kenderov and Moors [4] showed that if $(X, weak)$ is σfragmented by d then it must be σfragmented by the norm. Our next results go in this line.

Proposition 5. *If X is a Banach space with LSP, then X has $\mathcal{L}(\| \cdot \|, weak)$.*

Proposition 6. *If X is a Banach space such that $(X, weak)$ has dSLD, for a metric d finer than the weak topology, then $(X, weak)$ has $\| \cdot \|$SLD.*

By the Theorem of Mazur, it is easy to see that any Banach space always has span-$\mathcal{L}(\| \cdot \|, weak)$. What happens with the LSP on any Banach space?

Example 7. *Any Banach space of density character $\aleph_1$ has LSP. So if we assume CH, ℓ^∞ has the LSP, therefore LSP does not imply the SLD property.*

Definition 8. *Let X be a Banach space and $A \subset X^*$. We shall say that A has property $^*LSP$ if A has $\mathcal{L}(\| \cdot \|, w^*)$. (Analagously span$^*LSP$.)*

If X is a Banach space, we shall denote by B_{X^*} its closed dual unit ball. We show the following:

Proposition 9. *Let X be a Banach space such that either X^* has span$^*LSP$ or B_{X^*} has $^*LSP$, then X is Asplund.*

In our next result we characterize the WCG and Asplund Banach spaces.

Theorem 10. *Let X be a Banach space. The following conditions are equivalent: i) X is WCG and Asplund;*
ii) B_{X^*} has $^*LSP$;
iii) X^* $^*LSP$.

The main techniques used in proving Proposition 9 and Theorem 10, are the construction of Projectional Generators (see [8]) in order to obtain shrinking PRI. Indeed, the map: $\Phi : B_{X^*} \to 2^X$, defined as follows is the projectional generator, defined in the whole dual ball, we are looking for: for $x^* \in B_{X^*}$ set $\Psi(x^*)$ to be a countable subset of X norming x^* and consider $S(x^*)$ the countably valued map given by $^*LSP$. Finally define $\Phi(x^*) = \Psi(S(x^*))$.

The topological version of the former theorem will give us a characterization of Eberlein compact. It is well known Rosenthal's internal characterization of the class of Eberlein compacta. It is also clear the relationship between Eberlein compact spaces with the existence of a continuous injection into a $c_0(\Gamma)$ space. In our next result we give another internal characterization of Eberlein compact spaces.

Theorem 11. *Let (K, τ) be a compact Hausdorff space. K is Eberlein compact if, and only if, there exists a taulower semicontinuous metric d, such that K has $\mathcal{L}(d, \tau)$.*

Remark 1. *The proof of Theorem 11 doesn't follow from the Banach space versions. It also makes use of PRI but in a more complicated way.*

In the line of Namioka's question we can prove the following

Theorem 12. *Let K be a RN compact. Then, K is Eberlein compact if, and only if, K has LSP.*

Indeed, the Banach space version of Theorem 12 is the following

Proposition 13. *Let X be an Asplund generated Banach space, i.e. (there exists an Asplund space E and a map $T : E \to X$ with $\overline{T(E)}^{\| \cdot \|} = X$). Then X is WCG if, and only if, (B_{X^*}, w^*) has LSP.*

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